

Strong geodetic problem in grid like architectures

Sandi Klavžar ^{a,b,c}

Paul Manuel ^d

January 29, 2018

^a Faculty of Mathematics and Physics, University of Ljubljana, Slovenia

`sandi.klavzar@fmf.uni-lj.si`

^b Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia

^c Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

^d Department of Information Science, College of Computing Science and Engineering,

Kuwait University, Kuwait

`pauldmanuel@gmail.com`

Abstract

A recent variation of the classical geodetic problem, the strong geodetic problem, is defined as follows. If G is a graph, then $sg(G)$ is the cardinality of a smallest vertex subset S , such that one can assign a fixed geodesic to each pair $\{x, y\} \subseteq S$ so that these $\binom{|S|}{2}$ geodesics cover all the vertices of G . In this paper, the strong geodetic problem is studied on Cartesian product graphs. A general upper bound is proved on the Cartesian product of a path with an arbitrary graph and showed that the bound is tight on thin grids and thin cylinders.

Keywords: geodetic problem; strong geodetic problem; Cartesian product of graphs; grids; cylinders

AMS Subj. Class.: 05C12, 05C70, 05C76; 68Q17

1 Introduction

Covering vertices of a graph with shortest paths is a problem that naturally appears in different applications; modelling them one arrives at different variations of the problem.

The classical *geodetic problem* [9] is to determine a smallest set of vertices S of a given graph such that the (shortest-path) intervals between them cover all the vertices. The investigation on this problem up to 2011 has been surveyed in [2], see also the book [18] for a general framework on the geodesic convexity. Recent developments on the geodetic problem include the papers [6, 7, 19], for a detailed literature survey see [14, 15]. Another variation of the shortest-path covering problem is the *isometric path problem* [8] where one is asked to determine the minimum number of geodesics required to cover the vertices, see also [17]. Yet another related concept is the Steiner (geodetic) number which generalizes the geodetic number from intervals between two vertices to Steiner intervals between largest vertex subsets. More precisely, the Steiner interval $I(S)$ of S is the union of all the vertices of G that belong to some Steiner tree for S , the smallest cardinality of a set S such that $I(S) = V(G)$ is the *Steiner (geodetic) number* of G . In [16] this invariant was denoted $\text{sg}(G)$, while in most papers it is denoted with $s(G)$, cf. [5, 10, 21], so it seems that using the notation $\text{sg}(G)$ in this paper does not present a conflict.

Motivated by applications in social networks, very recently the so called strong geodetic problem was introduced in [14] as follows. Let $G = (V, E)$ be a graph. Given a set $S \subseteq V$, for each pair of vertices $\{x, y\} \subseteq S$, $x \neq y$, let $\tilde{P}(x, y)$ be a *selected fixed* shortest path between x and y . Then we set

$$\tilde{I}(S) = \{\tilde{P}(x, y) : x, y \in S\},$$

and let $V(\tilde{I}(S)) = \bigcup_{\tilde{P} \in \tilde{I}(S)} V(\tilde{P})$. If $V(\tilde{I}(S)) = V$ for some $\tilde{I}(S)$, then the set S is called a *strong geodetic set*. The *strong geodetic problem* is to find a minimum strong geodetic set S of G . Clearly, the collection $\tilde{I}(S)$ of geodesics consists of exactly $\binom{|S|}{2}$ paths. The cardinality of a minimum strong geodetic set is the *strong geodetic number* of G and denoted by $\text{sg}(G)$. For an edge version of the strong geodetic problem see [15].

In [14] it was proved that the strong geodetic problem is NP-complete. Hence it is desirable to determine it on specific classes of graphs of wider interest. In this paper we follow this direction and proceed as follows. In the next section we first recall relevant properties of the Cartesian product of graphs. Afterwards we prove a lower bound on the strong geodetic number of Cartesian products in which one factor is a path. In Section 3 we demonstrate that the bound is tight for grids $P_r \square P_n$ and cylinders $P_r \square C_n$ for the case when r is large enough with respect to n ; we will roughly refer to such graphs as *thin grids* and *thin cylinders*, respectively. But first we define concepts needed.

All graphs considered in this paper are simple and connected. The *distance* $d_G(u, v)$ between vertices u and v of a graph G is the number of edges on a shortest u, v -path alias u, v -geodesic. The *diameter* $\text{diam}(G)$ of G is the maximum distance between the vertices of G . We will use the notation $[n] = \{1, \dots, n\}$ and the convention that $V(P_n) = [n]$ for any $n \geq 1$ as well as $V(C_n) = [n]$ for any $n \geq 3$, where the edges of P_n and of C_n are defined in the natural way.

The *Cartesian product* $G \square H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$, vertices (g, h) and (g', h') being adjacent if either $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$. If $h \in V(H)$, then the subgraph of $G \square H$ induced by the vertices of the form (x, h) , $x \in V(G)$, is isomorphic to G ; it is denoted with G^h and called a G -layer. Analogously H -layers are defined; if $g \in V(G)$, then the corresponding H -layer is denoted H^g . G -layers are also referred to as *horizontal layers* or, especially for grid as *rows*, while H -layers are *vertical layers* or *columns*.

2 A lower bound on $\text{sg}(G \square P_r)$

In this section we prove a lower bound on the strong geodetic number of Cartesian products in which one factor is a path.

The classical geodetic number was already well investigated in Cartesian products of graphs, see [1, 13, 20]. The obtained results appear (and are!) more general than the results we present in this paper for the strong geodetic number. We believe that a reason for this is that the strong geodetic problem seems intrinsically more difficult than the geodetic problem. For instance, it is straightforward to see that the geodetic number of the complete bipartite graph $K_{m,n}$ is equal to $\min\{m, n, 4\}$, see [9]. On the other hand, to determine the strong geodetic number of $K_{m,n}$ is very demanding. As shown in [12] this reduces to a non-linear optimization problem, and with quite some effort, it was proved that if $n \geq 6$, then

$$\text{sg}(K_{n,n}) = \begin{cases} 2 \left\lceil \frac{-1 + \sqrt{8n+1}}{2} \right\rceil; & 8n - 7 \text{ is not a perfect square,} \\ 2 \left\lceil \frac{-1 + \sqrt{8n+1}}{2} \right\rceil - 1; & 8n - 7 \text{ is a perfect square.} \end{cases}$$

The statement of this result and the fact that for $m \neq n$ the value of $\text{sg}(K_{m,n})$ is not yet known points out the difficulty of the strong geodetic number. The intuition for this is that in order to determine the geodetic number, one “only” needs to find an appropriate set of vertices, while for the strong geodetic number this is only half of the

job because after selecting an appropriate set of vertices one has to carefully determine specific geodesics among the pairs of vertices.

We start by recalling some facts about the Cartesian product, especially about its metric properties. The Cartesian product is an associative and commutative operation. More precisely, the latter assertion means that the graphs $G \square H$ and $H \square G$ are isomorphic. We will implicitly (and explicitly) use this fact several times. Recall also that $G \square H$ is connected if and only if both G and H are connected. Moreover, the metric structure of Cartesian product graphs is well-understood, see [11, Chapter 12]. Its basis is the following result that was independently discovered several times, cf. [11, Lemma 12.1].

Proposition 2.1 *If (g, h) and (g', h') are vertices of a Cartesian product $G \square H$, then*

$$d_{G \square H}((g, h), (g', h')) = d_G(g, g') + d_H(h, h').$$

If $(g, h) \in V(G \square H)$, then the projections $p_G : V(G \square H) \rightarrow V(G)$ and $p_H : V(G \square H) \rightarrow V(H)$ are defined with $p_G((g, h)) = g$ and $p_H((g, h)) = h$. The projections p_G and p_H can be extended such that they also map the edges of $G \square H$. More precisely, if $e = (g, h)(g', h) \in E(G \square H)$, then $p_G(e) = gg' \in E(G)$, and if $e = (g, h)(g, h') \in E(G \square H)$, then $p_H(e) = hh' \in E(H)$. Furthermore, we can also consider $p_G(X)$ and $p_H(X)$, where X is a subgraph of $G \square H$.

Proposition 2.1 together with the fact that if (g, h) and (g', h) are vertices of the same G -layer, then every geodesic between (g, h) and (g', h) lies completely in the layer (see the first exercise in [11, 12.3 Exercises]) implies the following.

Corollary 2.2 *Let P be a geodesic in $G \square H$. If $e = (g, h)(g', h) \in E(P)$, then e is the unique edge of P with $p_G(e) = gg'$. Moreover, $p_G(P)$ is a geodesic in G .*

Of course, by the commutativity of the Cartesian product, the assertions of Corollary 2.2 also hold for the projection of P on H .

After this preparation we can state the following technical lemma.

Lemma 2.3 *Let G and H be graphs, Ω be a minimum strong geodetic set of $G \square H$, and $\tilde{I}(\Omega)$ its corresponding set of geodesics. If $|V(H)| > \text{diam}(G) \binom{|\Omega|}{2} + |\Omega|$, then there exists a G -layer G^h such that*

$$(i) \ E(G^h) \cap \left(\bigcup_{P \in \tilde{I}(\Omega)} E(P) \right) = \emptyset \text{ and}$$

(ii) $V(G^h) \cap \Omega = \emptyset$.

Proof. Let t be the number of G -layers with the property that none of their edges lies on some path from $\tilde{I}(\Omega)$, that is,

$$t = \left| \left\{ h \in V(H) : E(G^h) \cap \left(\bigcup_{P \in \tilde{I}(\Omega)} E(P) \right) = \emptyset \right\} \right|.$$

Let P be a geodesic from $\tilde{I}(\Omega)$. Consider the set of edges $p_G(E(P))$ of P that lie in G -layers. If $e, e' \in p_G(E(P))$, then $p_G(e) \neq p_G(e')$ by Corollary 2.2, that is, e and e' project on different edges of G . It follows that the edges of P lie in at most $\text{diam}(G)$ number of different G -layers. Hence, since $|\tilde{I}(\Omega)| = \binom{|\Omega|}{2}$, the number of G -layers that contain edges of the paths from $\tilde{I}(\Omega)$ is at most $\text{diam}(G) \binom{|\Omega|}{2}$. Consequently, because we have assumed that $|V(H)| > \text{diam}(G) \binom{|\Omega|}{2} + |\Omega|$ and as $|V(H)|$ is just the number of G -layers, we infer that $t > |\Omega|$. Therefore, by the pigeon-hole principle there exists at least one G -layer G^h , such that $E(G^h) \cap \left(\bigcup_{P \in \tilde{I}(\Omega)} E(P) \right) = \emptyset$ and $V(G^h) \cap \Omega = \emptyset$ as claimed. \square

We now restrict to Cartesian products of the form $G \square P_r$. Since we have assumed that $V(P_n) = [n]$, the G -layers of $G \square P_r$ are thus denoted with G^1, \dots, G^r . See Fig. 1 for a graph G , the Cartesian product $G \square P_4$, and the four G -layers.

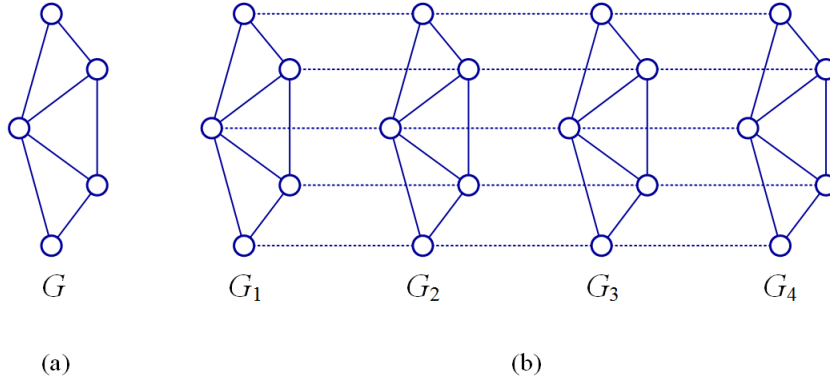


Figure 1: (a) Factor G . (b) $G \square P_4$, where the dotted edges are the edges of the P_4 -layers, the other edges belong to the G -layers G^1, G^2, G^3, G^4 .

The main result of this section reads as follows. (It will be applied a couple of times in the subsequent section.)

Lemma 2.4 *Let Ω be a minimum strong geodetic set of $G \square P_r$. If $r > \text{diam}(G) \binom{|\Omega|}{2} + |\Omega|$, then $\text{sg}(G \square P_r) \geq \lceil 2\sqrt{|V(G)|} \rceil$.*

Proof. Applying Lemma 2.3 we infer that $G \square P_r$ contains a G -layer, say G^i , such that no edge of G^i lies on a path from $\tilde{I}(\Omega)$ and such that $V(G^i) \cap \Omega = \emptyset$. Note that $i \neq 1$ and $i \neq r$, for otherwise the vertices of G^1 (resp. G^r) would not be covered with the paths from $\tilde{I}(\Omega)$. We can hence partition Ω into non-empty sets Ω_1 and Ω_2 by setting

$$\Omega_1 = \Omega \cap \left(\bigcup_{j=1}^{i-1} G^j \right) \quad \text{and} \quad \Omega_2 = \Omega \cap \left(\bigcup_{j=i+1}^r G^j \right),$$

cf. Fig. 2.

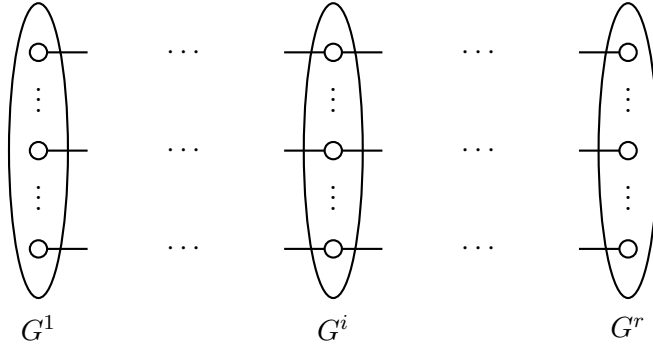


Figure 2: The G -layer G^i guaranteed by Lemma 2.3 is neither the first nor the last G -layer.

In order to cover the vertices of G^i , we must have $|\tilde{I}(\Omega)| \geq |V(G)|$. This in turn implies that $|\Omega_1| \cdot |\Omega_2| \geq |V(G)|$. Since the arithmetic mean is not smaller than the geometric mean, we have

$$\frac{|\Omega|}{2} = \frac{|\Omega_1| + |\Omega_2|}{2} \geq \sqrt{|\Omega_1| \cdot |\Omega_2|} \geq \sqrt{|V(G)|}.$$

Thus $|\Omega| \geq 2\sqrt{|V(G)|}$. Since the number of vertices is an integer we conclude that $\text{sg}(G \square P_r) \geq \lceil 2\sqrt{|V(G)|} \rceil$. \square

3 Thin grids and cylinders

In this section we determine the strong geodetic number of thin grids $P_r \square P_n$ ($r \gg n$) and thin cylinders $P_r \square C_n$ ($r \gg n$). (The geodetic number in Cartesian products was

investigated in [1, 19], while in [4] and in [3] it was studied on strong products and lexicographic products, respectively.) Recall that by our convention on the vertex sets of paths and cycles, $V(P_r \square P_n) = V(P_r \square C_n) = \{(i, j) : i \in [r], j \in [n]\}$.

Lemma 3.1 *If $2 \leq n \leq r$, then $\text{sg}(P_r \square P_n) \leq \lceil 2\sqrt{n} \rceil$.*

Proof. In order to prove the inequality, we need to construct a strong geodetic set of cardinality $\lceil 2\sqrt{n} \rceil$.

We first consider the case when n is a perfect square, $n = k^2$. For each $i \in [k]$ define the vertices a_i and b_i of $P_r \square P_n$ with

$$\begin{aligned} a_i &= (1, (i-1)k + i), \\ b_i &= (r, (i-1)k + i), \end{aligned}$$

and set $S = \{a_1, a_2, \dots, a_k\} \cup \{b_1, b_2, \dots, b_k\}$. Note that $a_1 = (1, 1)$, $b_1 = (r, 1)$, $a_k = (1, k^2)$, and $b_k = (r, k^2)$ are the four vertices of $P_r \square P_n$ of degree 2 and that $|S| = 2k = 2\sqrt{n}$. Now we show that S is a strong geodetic set of $P_r \square P_n$ by constructing $\tilde{I}(S)$ such that $V(\tilde{I}(S)) = V(P_r \square P_n)$. It will suffice to select a geodesic for each pair of vertices a_i and b_j to achieve our goal.

For each $i \in [k]$ there is a unique a_i, b_i -geodesic which thus must belong to $\tilde{I}(S)$. We next inductively add geodesics to $\tilde{I}(S)$. First, add geodesics to $\tilde{I}(S)$ one by one from a_1 to respectively b_2, \dots, b_k as follows. Start in a_1 and traverse the first column until the first not yet traversed row is reached. Then traverse this row and complete the path by traversing the last column until the vertex b_j that is just considered is reached. These paths thus cover $k-1$ rows. Then proceed along the same way for the vertices a_2, \dots, a_{k-1} , respectively covering $k-2, \dots, 1$ new rows. Repeat next the above procedure by inductively construction the geodesics from a_k, \dots, a_2 to the vertices b_j . In this way the remaining rows are covered. The construction is illustrated in Fig. 3.

Assume next that $n = k^2 + \ell$, where $1 \leq \ell \leq k$. In this case $\lceil 2\sqrt{n} \rceil = 2k + 1$. Define the vertices a_i and b_i as above and set

$$S = \{a_1, a_2, \dots, a_k\} \cup \{b_1, b_2, \dots, b_k\} \cup \{(1, n)\}.$$

As in the previous case, all the vertices of the subgraph $P_r \square P_n$ can be covered using the vertices from $\{(1, i^2), (r, i^2) : i \in [k]\}$. Then it is not difficult to show that the vertex $(1, n)$ will take care of the remaining vertices of $P_r \square P_n$. Since $|S| = 2k + 1$, we are done also in this case.

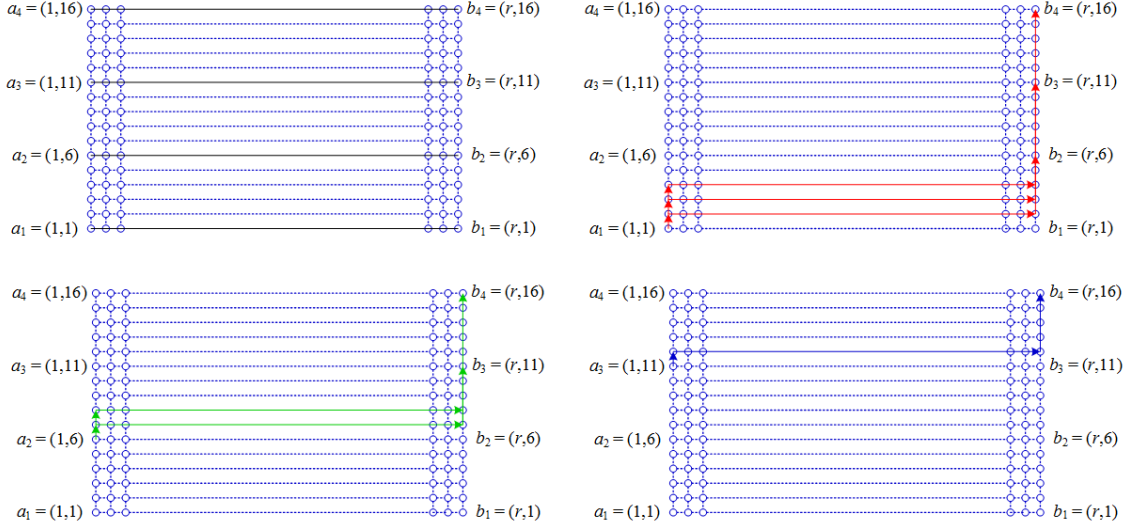


Figure 3: $P_r \square P_{4^2}$ and the geodesics from $\tilde{I}(S)$ between a_i and b_j , where $i < j$.

Finally, suppose that $n = k^2 + \ell$, where $k+1 \leq \ell \leq 2k$. In this case $\lceil 2\sqrt{n} \rceil = 2k+2$. Setting

$$S = \{a_1, a_2, \dots, a_k\} \cup \{b_1, b_2, \dots, b_k\} \cup \{(1, n), (r, n)\}$$

we can argue similarly as above that S is a strong geodetic set of $P_r \square P_n$. \square

The first main result of this section reads as follows.

Theorem 3.2 *If $r > \binom{\lceil 2\sqrt{n} \rceil}{2}(n-1) + \lceil 2\sqrt{n} \rceil$, then $\text{sg}(P_r \square P_n) = \lceil 2\sqrt{n} \rceil$.*

Proof. Let Ω be a minimum strong geodetic set of $P_r \square P_n$. By Lemma 3.1 we know that $|\Omega| \leq \lceil 2\sqrt{n} \rceil$. Hence

$$r > \binom{\lceil 2\sqrt{n} \rceil}{2}(n-1) + \lceil 2\sqrt{n} \rceil \geq \binom{|\Omega|}{2}(n-1) + |\Omega|.$$

As $\text{diam}(P_n) = n-1$, Lemma 2.4 implies that $\text{sg}(P_r \square P_n) = |\Omega| \geq \lceil 2\sqrt{n} \rceil$. \square

Of course, it would be desirable to determine the exact strong geodetic number for all grids. To see that Theorem 3.2 cannot be extended to all grids consider the product $P_7 \square P_7$. In Fig. 4 we have produced a strong geodetic set consisting of 5 vertices. Thus $\text{sg}(P_7 \square P_7) \leq 5 < \lceil 2\sqrt{7} \rceil = 6$.

In the rest of the section we consider cylinders $P_r \square C_n$ and prove an analogous result for thin cylinders as we did for thin grids. We start with:

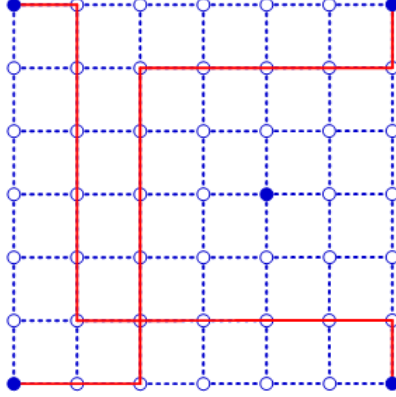


Figure 4: The blue bullets form a strong geodetic set of $P_7 \square P_7$. For the sake of clarity, only two geodesics are drawn. The reader can easily identify other geodesics.

Lemma 3.3 *If $2 \leq n \leq r$, then $\text{sg}(P_r \square C_n) \leq \lceil 2\sqrt{n} \rceil$.*

Proof. The proof of Lemma 3.1 is modified to accommodate cylinders. We first consider the case when n is a perfect square, $n = k^2$. For each $i \in [k]$ define the vertices a_i and b_i of $P_r \square C_n$ with

$$\begin{aligned} a_i &= (1, (i-1)k + 1), \\ b_i &= (r, (i-1)k + 1), \end{aligned}$$

and set $S = \{a_1, a_2, \dots, a_k\} \cup \{b_1, b_2, \dots, b_k\}$. We claim that S is a strong geodetic set of $P_r \square C_n$ by constructing $\tilde{I}(S)$ such that $V(\tilde{I}(S)) = V(P_r \square C_n)$.

Assume first that k is odd. Select the geodesics between a_1 and $b_2, b_3, \dots, b_{\lfloor k/2 \rfloor}$ and geodesics between a_2 and $b_1, b_k, \dots, b_{\lfloor k/2 \rfloor + 2}$ such that all the P_r -layers P_r^2, \dots, P_r^{k-1} are covered by them. See Figs. 5 and 6, where the case $P_2 \square C_{25}$ is illustrated; that is, $r = 2$ and $k = 5$.

By symmetry we can design geodesics starting from a_i and a_{i+1} such that all the corresponding P_k -layers are covered. In conclusion, S is a strong geodetic set. If k is even, we can proceed similarly to verify that S is a strong geodetic set also in this case. Finally, if $n = k^2 + \ell$ and $1 \leq \ell \leq 2k$, then by a similar construction as in the proof of Lemma 3.1 (by adding either one or two new vertices to S , depending on whether $1 \leq \ell \leq k$ or $k+1 \leq \ell \leq 2k$, the construction is completed. \square

We are now ready for the second main result of this section. It will be deduced similarly as Theorem 3.2.

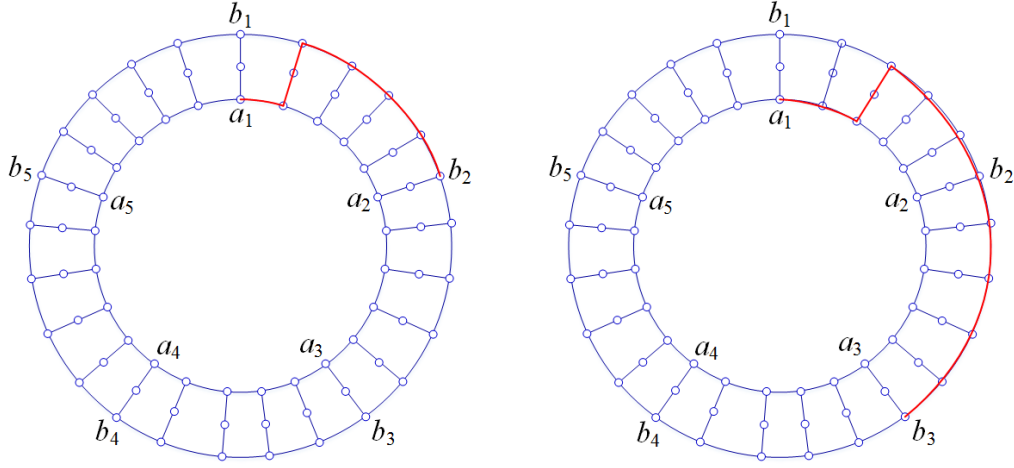


Figure 5: The paths marked by red edges are members of $\tilde{I}(S)$. The first geodesic is between a_1 and b_2 and the second geodesic is between a_1 and b_3 .

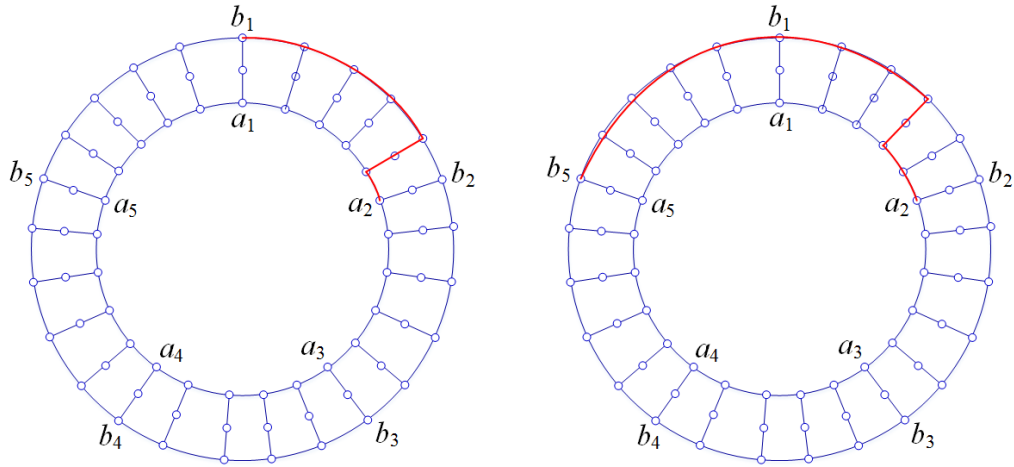


Figure 6: The paths marked by red edges are members of $\tilde{I}(S)$. The first geodesic is between a_2 and b_1 and the second geodesic is between a_2 and b_5 .

Theorem 3.4 *If $r > \binom{\lceil 2\sqrt{n} \rceil}{2} \lfloor \frac{n}{2} \rfloor + \lceil 2\sqrt{n} \rceil$, then $\text{sg}(P_r \square C_n) = \lceil 2\sqrt{n} \rceil$.*

Proof. Let Ω be a minimum strong geodetic set of $P_r \square C_n$. From Lemma 3.3 we know that $|\Omega| \leq \lceil 2\sqrt{n} \rceil$. Hence

$$r > \binom{\lceil 2\sqrt{n} \rceil}{2} \left\lfloor \frac{n}{2} \right\rfloor + \lceil 2\sqrt{n} \rceil \geq \binom{|\Omega|}{2} \left\lfloor \frac{n}{2} \right\rfloor + |\Omega|.$$

As $\text{diam}(C_n) = \lfloor \frac{n}{2} \rfloor$, Lemma 2.4 implies that $\text{sg}(P_r \square C_n) = |\Omega| \geq \lceil 2\sqrt{n} \rceil$. \square

4 Further research

In this paper we have studied the strong geodetic problem on Cartesian product graphs and determined the strong geodetic number for “thin” grids and cylinders. The first natural problem is of course to determine the strong geodetic number for all grids and cylinders. Next it would be interesting to consider the strong geodetic number on additional interesting Cartesian product graphs, such as torus graphs (product of two cycles) as well as on general Cartesian products. More generally, we can ask for the strong geodetic number on Cartesian product of more than two factors, in particular on multidimensional grid graphs.

Acknowledgement

S.K. acknowledges the financial support from the Slovenian Research Agency (research core funding No. P1-0297).

References

- [1] B. Brešar, S. Klavžar, A. Tepeh Horvat, On the geodetic number and related metric sets in Cartesian product graphs, *Discrete Math.* 308 (2008) 5555–5561.
- [2] B. Brešar, M. Kovše, A. Tepeh, Geodetic sets in graphs, in: *Structural Analysis of Complex Networks*, Birkhäuser/Springer, New York (2011) 197–218.
- [3] B. Brešar, T. Kraner Šumenjak, A. Tepeh, The geodetic number of the lexicographic product of graphs, *Discrete Math.* 311 (2011) 1693–1698.
- [4] J. Cáceres, C. Hernando, M. Mora, I.M. Pelayo, M.L. Puertas, On the geodetic and the hull numbers in strong product graphs, *Comput. Math. Appl.* 60 (2010) 3020–3031.

- [5] G. Chartrand, P. Zhang, The Steiner number of a graph, *Discrete Math.* 242 (2002) 41–54.
- [6] T. Ekim, A. Erey, Block decomposition approach to compute a minimum geodetic set, *RAIRO Oper. Res.* 48 (2014) 497–507.
- [7] T. Ekim, A. Erey, P. Heggernes, P. van't Hof, D. Meister, Computing minimum geodetic sets in proper interval graphs, *Lecture Notes Comp. Sci.* 7256 (2012) 279–290.
- [8] D. C. Fisher, S. L. Fitzpatrick, The isometric path number of a graph, *J. Combin. Math. Combin. Comput.* 38 (2001) 97–110.
- [9] F. Harary, E. Loukakis, C. Tsouros, The geodetic number of a graph, *Math. Comput. Modelling* 17 (1993) 89–95.
- [10] C. Hernando, T. Jiang, M. Mora, I. M. Pelayo, C. Seara, On the Steiner, geodetic and hull numbers of graphs, *Discrete Math.* 293 (2005) 139–154.
- [11] W. Imrich, S. Klavžar, D. F. Rall, *Topics in Graph Theory: Graphs and Their Cartesian Product*, A K Peters, Ltd., Wellesley, MA, 2008.
- [12] V. Iršič, Strong geodetic number of complete bipartite graphs and of graphs with specified diameter, *arXiv:1708.02416 [math.CO]*.
- [13] T. Jiang, I. Pelayo, D. Pritikin, Geodesic convexity and Cartesian products in graphs, manuscript, 2004, available at http://jupiter.math.nctu.edu.tw/~weng/references/others/graph_product_2004.pdf.
- [14] P. Manuel, S. Klavžar, A. Xavier, A. Arokiaraj, E. Thomas, Strong geodetic problem in networks: computational complexity and solution for Apollonian networks, *arXiv:1708.03868 [math.CO]*.
- [15] P. Manuel, S. Klavžar, A. Xavier, A. Arokiaraj, E. Thomas, Strong edge geodetic problem in networks, *Open Math.* 15 (2017) 1225–1235.
- [16] O. R. Oellermann, M. L. Puertas, Steiner intervals and Steiner geodetic numbers in distance-hereditary graphs, *Discrete Math.* 307 (2007) 88–96.
- [17] J.-J. Pan, G. J. Chang, Isometric path numbers of graphs, *Discrete Math.* 306 (2006) 2091–2096.

- [18] I.M. Pelayo, Geodesic convexity in graphs, Springer Briefs in Mathematics, Springer, New York, 2013.
- [19] J. A. Soloff, R. A. Márquez, L. M. Friedler, Products of geodesic graphs and the geodetic number of products, *Discuss. Math. Graph Theory* 35 (2015) 35–42.
- [20] Y. Ye, Z. Sheng, Q. Ming, Y. H. Mo, Geodetic numbers of Cartesian products of trees, *Acta Math. Appl. Sin.* 31 (2008) 514–519.
- [21] I. G. Yero, J. A. Rodríguez-Velázquez, Analogies between the geodetic number and the Steiner number of some classes of graphs, *Filomat* 29 (2015) 1781–1788.